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Pricing Infinite Horizon Programs*

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1. INTRODUCTION

This paper continues the earlier studies [17, 18] whose object was to develop an abstract framework for the analysis of problems of resource allocation in continuous time over an infinite horizon. These papers gave conditions on the preferences and technology sufficient to guarantee the existence of an optimal program. In this paper I shall provide an answer to the following question. *What strengthening of these conditions guarantees the existence of supporting prices?*

I pose the problem of resource allocation as a convex programming problem in a suitable infinite dimensional space (Section 2). A convex programming problem is characterised by a family of preferred sets induced by a utility function U and a technologically feasible set $\mathscr{F}$. Such a framework is not confined to the analysis of an economy with a single representative agent. If the economy consists of a finite number of consumers, with preference orderings representable by concave increasing utility functions $(U_1,..., U_k)$, and a finite number of producers, with technology sets $(\mathscr{F}_1,..., \mathscr{F}_m)$, then we may let $U = \sum_{j=1}^{k} \alpha_j U_j$, where $\alpha_j \geqq 0$, $\sum_{j=1}^{k} \alpha_j = 1$, and $\mathscr{F} = \sum_{i=1}^{m} \mathscr{F}_i$. Suppose now that the existence of an optimal program and an associated supporting price is established for each parameter value α in the simplex. Under certain conditions (see e.g., Section 5, Theorem 5.15) such a pair leads to an equilibrium with transfer payments. A fixed point argument involving the parameter α and the transfer payments can then be added to ensure the existence of a competitive equilibrium [6, 16, 20]. In this way an *equilibrium* (allocation-price) emerges as an *optimal program* and an associated *supporting price* for a particular parameter value in a family of convex programming problems.

When the problem of resource allocation ceases to be finite dimensional, an essential step in the analysis is the choice of an appropriate *program* (*commodity*) *space*. In [17] I considered a weighted version of the space of

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Lebesgue integrable functions ($\mathscr{L}_\nu^{1,n}$) where the weight ν is determined by the rate at which technologically feasible programs can grow over the infinite horizon. In this paper I shall consider the subspace $W_\nu^{1,n} \subset \mathscr{L}_\nu^{1,n}$ of *absolutely continuous functions* x for which both x and its derivative $\dot{x}$ lie in $\mathscr{L}_\nu^{1,n}$. If $\mathscr{L}_{\nu^{-1}}^{\infty,n}$ denotes a weighted version of the space of bounded measurable functions, then prices will be chosen from the subspace $W_{\nu^{-1}}^{\infty,n}$ consisting of the *absolutely continuous functions* p for which both p and its derivative $\dot{p}$ lie in $\mathscr{L}_{\nu^{-1}}^{\infty,n}$. Such a choice of price-quantity spaces allows one to both price programs over an infinite horizon and cope with the special relation (absolute continuity) that exists between stocks and flows in a continuous time analysis. These two spaces have the further important property that a certain *adjoint relation* (*integration by parts* and the *transversality condition*) is valid (Section 3).

In the classical theorem of welfare economics of Arrow and Debreu [8, Theorem 2], if the technologically feasible set has a nonempty interior, then the existence of prices follows directly from a separation theorem. The feasible set $\mathscr{F}$ for the convex programming problem of Section 2 has however a more complex structure. Thus, while a separation theorem can be applied to obtain the existence of prices, a direct application of such a theorem by failing to take into account the linear operations involved in the construction of the feasible set fails to reveal the important induced adjoint relation between the prices. These linear operator relations between the quantities and the induced adjoint operator relations between the prices are basic to the content of the true support price relation, the *Euler–Lagrange inclusion* ($\mathscr{E}$ in Section 4). Thus, a different approach is required.

When the sets that characterise the utility function U and the technology set $\mathscr{F}$ are closed convex sets, a useful general framework has emerged from the work of Rockafellar [10, Chapter III]. The approach involves the analysis of a *family of perturbations* of the original problem. A certain *marginal function* is introduced whose superdifferentiability at the origin is equivalent to the existence of supporting prices (Theorem 4.7). The standard criterion for the superdifferentiability of the marginal function is the *Slater condition* [10, Eq. 5.24]. In Section 4 I give a natural generalisation of this condition applicable to the convex programming problem of Section 2 (Theorem 4.10). This leads to the basic criterion for the existence of prices, that *the feasible set be nonempty under a family of perturbations* (Theorem 4.11). It is easy to see that this criterion includes the usual constraint qualification condition in the standard Kuhn–Tucker theorem [10, Theorem 5.2].

In Section 5 the results of Sections 2–4 are applied to establish the existence of supporting prices for a continuous time infinite horizon economy. The technology and preferences are assumed to be representable by integral functionals along the lines of [17]. I show that if the technology and preferences are regular and satisfy compatible growth conditions, then there exists an optimal program (Theorem 5.12(i)). If in addition the technology has a property of separability and if consumption and investment are bounded below on an optimal path, then there exists a supporting price (Theorem 5.12(ii)). The existence of an equilibrium with transfer payments, in an economy with a finite number of agents, is established in Theorem 5.15. As noted in Remark 5.18, by supplementing this result with an appropriate fixed point argument we obtain the existence of a competitive equilibrium. It then follows from Theorem 5.12(iii) that the resulting *equilibrium* (allocation-price) may be characterised as a solution of the local *Hamiltonian inclusions* (H, T). An objective sought by Cass and Shell [7] of writing the equilibrium of a competitive dynamical system as the solution of a system of Hamiltonian equations is thus attained.

The paper concludes in Section 6 with two simple applications of the results of Section 5. In the first example feasible programs are *bounded*; in the second example feasible programs are *unbounded* and capable of growth at a positive exponential rate.

Existence and duality in infinite dimensional spaces have been treated extensively in the literature. A useful background is provided by Chapters I–III of Ekeland and Teman [10]. In Section 4, I assume that the reader is familiar with two important results: first, the way in which a family of *perturbations* of the problem $(\mathscr{P})$ leads via the theory of *conjugate convex functions* to the associated dual problem $(\mathscr{P}^*)$; second, the way in which the *Fenchel conjugacy relation* $(\mathscr{N})$ between $(\mathscr{P})$ and $(\mathscr{P}^*)$ leads via the *marginal function* (4.5) to the basic criterion for the existence of supporting prices (Theorem 4.7). These results are given in [10, Chapter III].

A systematic treatment of existence and duality for a closely related class of problems in the finite horizon case has been given by Rockafellar [22]. This paper has been strongly influenced by the results of Aubin–Clarke [3] which also treats convex programming problems in which the utility function U is not necessarily convex. At the time that this paper was completed the results of Araujo–Scheinkman [1] came to my notice. They give conditions for a related class of problems for the existence of prices, when the program space is the space of bounded measurable functions, and hence provide a natural link with the work of Bewley [5].

2. Abstract Form of Resource Allocation Problem

The class of resource allocation problems that I shall consider can be reduced to a *convex programming problem* in a suitable infinite dimensional space. Let (Z, W) denote Banach spaces, where W a dense subset of Z denotes the *program space*. I assume that choice among programs is determined by a *preference ordering* $\succsim$ which is representable by a utility function $U(\cdot)$ satisfying

ASSUMPTION 1. *$U(\cdot): Z \to (-\infty, \infty)$ is a continuous concave function.*

To characterise the technology I introduce two additional Banach spaces (V, Q), two feasible subsets $\mathscr{A} \subset Z$, $\mathscr{B} \subset Q$, and a technology correspondence $G(\cdot): Z \to V$ with graph $\mathscr{G} = \{(z, y) \mid y \in G(z)\}$ satisfying

ASSUMPTION 2. *$\mathscr{A}$, $\mathscr{B}$, and $\mathscr{G}$ are closed convex sets and* $\operatorname{dom} G = \{z \in Z \mid G(z) \neq \varnothing\} \supset \mathscr{A}$.

I introduce a *differential operator* L and a *boundary operator* λ and let $\Lambda z = (Lz, \lambda z)$ for $z \in W$. Let $\mathscr{L}(W, V)$ denote the *continuous linear operators* from W to V.

ASSUMPTION 3. $L \in \mathscr{L}(W, V)$, $\lambda \in \mathscr{L}(W, Q)$.

If I introduce the correspondence $K(\cdot): Z \to V \times Q$ defined by

$$K(z) = G(z) \times \mathscr{B}, \qquad z \in Z, \tag{2.1}$$

then the set of *technologically feasible programs* $\mathscr{F}$ will be given by

$$\mathscr{F} = \{z \in \mathscr{A} \mid 0 \in K(z) - \Lambda z\}. \tag{2.2}$$

2.1 DEFINITION. I say that $(U, \mathscr{F})$ are *regular* if Assumptions 1–3 are satisfied.

2.2 DEFINITION. A program $\bar{z} \in \mathscr{F}$ that solves the problem

$$\sup_{z \in \mathscr{F}} U(z) \tag{$\mathscr{P}$}$$

will be called an *optimal program*.

3. PRICE-QUANTITY SPACES AND LINEAR OPERATORS

In this section I shall begin to reduce the abstract problem of the previous section to a more concrete form. I shall introduce the infinite dimensional spaces appropriate for the analysis of problems of resource allocation in continuous time over an infinite horizon.

Let $\mathscr{M}^n = \mathscr{M}(0, \infty; R^n)$ denote the space of Lebesgue measurable functions defined on $[0, \infty)$ with values in R^n, $n \geq 1$. I shall be concerned with certain linear subspaces of $\mathscr{M}^n$.

3.1 DEFINITION. A function $\gamma \in \mathscr{M}$ satisfying $0 < \gamma(t) < \infty$ a.e. will be called a *growth function.* A function $\nu \in \mathscr{M}$ satisfying

$$0 < \nu(t) < \infty \quad \text{a.e.,} \qquad \int_0^\infty \nu(t)\, dt < \infty$$

will be called a *density function* for $[0, \infty)$. Let γ be a growth function. I say that ν is a *density function relative to* γ if ν is a density function satisfying

$$\int_0^\infty \nu(t)\, \gamma(t)\, dt < \infty.$$

Let γ be a growth function (to be determined in the subsequent analysis) and let ν be a density function relative to γ. For the quantities I am led to consider the following weighted space of Lebesgue integrable functions

$$\mathscr{L}_\nu^{1,n} = \left\{ x \in \mathscr{M}^n \;\middle|\; \|x\|_1 = \int_0^\infty \|x(t)\|\, \nu(t)\, dt < \infty \right\}.$$

For $x \in \mathscr{L}_\nu^{1,n}$ we can define its *distributional derivative* $\dot{x}$ [27, p. 49] and we are led to consider the space

$$W_\nu^{1,n} = \{ x \in \mathscr{L}_\nu^{1,n} \mid \dot{x} \in \mathscr{L}_\nu^{1,n} \}.$$

It is readily shown that this space coincides with the space of *absolutely continuous* functions $x \in \mathscr{L}_\nu^{1,n}$ whose derivative $\dot{x}$ also lies in $\mathscr{L}_\nu^{1,n}$ [25, Theorem III, p. 55]. $W_\nu^{1,n}$ is a Banach space under the norm

$$|||x|||_1 = \int_0^\infty (\|x(t)\| + \|\dot{x}(t)\|)\, \nu(t)\, dt. \tag{3.1}$$

For the prices I consider the following weighted space of essentially bounded measurable functions

$$\mathscr{L}_{\nu^{-1}}^{\infty,n} = \left\{ p \in \mathscr{M}^n \;\middle|\; \|p\|_\infty = \operatorname*{ess\,sup}_{0 < t < \infty} \frac{1}{\nu(t)} \|p(t)\| < \infty \right\},$$

where $\operatorname{ess\,sup}_{0<t<\infty} \|p(t)\| = \inf\{\alpha \in R \mid \lambda(t \mid \|p(t)\| > \alpha) = 0\}$, where $\lambda(A)$ denotes the Lebesgue measure of the set A. For $p \in \mathscr{L}_{\nu^{-1}}^{\infty,n}$ we can define its distributional derivative $\dot{p}$ and are thus led to consider the space

$$W_{\nu^{-1}}^{\infty,n} = \{ p \in \mathscr{L}_{\nu^{-1}}^{\infty,n} \mid \dot{p} \in \mathscr{L}_{\nu^{-1}}^{\infty,n} \}$$

which is a Banach space under the norm

$$\|\|p\|\|_\infty = \operatorname*{ess\,sup}_{0<t<\infty} \frac{1}{v(t)} (\|p(t)\| + \|\dot{p}(t)\|).$$

$W^{\infty;n}_{v^{-1}}$ is just the space of *absolutely continuous* functions $p \in \mathscr{L}^{\infty;n}_{v^{-1}}$ whose derivative $\dot{p}$ also lies in $\mathscr{L}^{\infty;n}_{v^{-1}}$. For $x \in \mathscr{L}^{1,n}_{v}$ and $p \in \mathscr{L}^{\infty;n}_{v^{-1}}$ I define the *scalar product*

$$\langle p, x\rangle = \int_0^\infty p(t)\, x(t)\, dt$$

noting that

$$|\langle p, x\rangle| \leqq \int_0^\infty \|p(t)\| \frac{1}{v(t)} \|x(t)\|\, v(t)\, dt \leqq \|p\|_\infty \|x\|_1 < \infty.$$

The following lemma on *integration by parts* is basic to the analysis that follows.

3.1 LEMMA. *If* $x \in W^{1,n}_v$, $p \in W^{\infty;n}_{v^{-1}}$, *then* (i) $\lim_{t\to\infty} p(t)\, x(t) = 0$ *and* (ii) $\int_0^\infty p(t)\, \dot{x}(t)\, dt + \int_0^\infty \dot{p}(t)\, x(t)\, dt + p(0)\, x(0) = 0$.

Proof. Since $x \in W^{1,n}_v$, $p \in W^{\infty;n}_{v^{-1}}$, we have

$$\int_0^\infty |p(t)\, x(t)|\, dt \leqq \|p\|_\infty \|x\|_1 < \infty \tag{3.2}$$

$$\int_0^\infty \left|\frac{d}{dt} p(t)\, x(t)\right| dt \leqq \|\dot{p}\|_\infty \|x\|_1 + \|p\|_\infty \|\dot{x}\|_1 < \infty. \tag{3.3}$$

Equation (3.2) implies $p(t)\, x(t) \to \eta$ as $t \to \infty$ for some $\eta \in R$; (3.3) implies $\eta = 0$. Since x and p are absolutely continuous, the function ϕ defined by $\phi(t) = p(t)\, x(t)$ a.e. is absolutely continuous. Thus by the theorem on absolutely continuous functions [24, Corollary 14, p. 107] for any $0 \leqq t < n < \infty$, $\phi(n) = \phi(t) + \int_t^n \phi'(\tau)\, d\tau$. Let $\phi_n'(\tau) = \chi_{[0,n]}(\tau)\, \phi'(\tau)$, then (i) implies $0 = \phi(t) + \lim_{n\to\infty} \int_t^\infty \phi_n'(\tau)\, d\tau$. Since $\phi_n'(t) \to \phi'(\tau)$ a.e. as $n \to \infty$ and since $|\phi_n'(\tau)| \leqq |\phi'(\tau)|$ a.e., (3.3) and the dominated convergence theorem [24, Theorem 15, p. 88] imply $\lim_{n\to\infty} \int_t^\infty \phi_n'(\tau)\, d\tau = \int_t^\infty \phi'(\tau)\, d\tau$ so that $\phi(t) = -\int_t^\infty \phi'(\tau)\, d\tau$, $\forall t \in [0, \infty)$. (ii) follows with $t = 0$. ∎

I am led to consider the following choice of spaces for the problem in Section 2. For integers $n \geqq 1$, $m \geqq 0$

$$Z = \mathscr{L}^{1,n+m}_v, \qquad W = W^{1,n}_v \times \mathscr{L}^{1,m}_v, \qquad V = \mathscr{L}^{1,n}_v, \qquad Q = R^n. \tag{3.4}$$

I let $z=(x,c)$ where x is a path of stocks of n capital goods and c is a path denoting consumption and other flow activity for m commodities. The differential operator L and the boundary operator λ are defined by

$$Lz=\frac{d}{dt}(x)=\dot{x}, \qquad \lambda z=x(0), \qquad \forall z=(x,c)\in W, \tag{3.5}$$

where $\dot{x}$ is the *path of investment* associated with x and $x(0)$ is its associated vector of *initial stocks*.

We need to find a *pair of linear operators* suitable for characterising a *dual problem* from which supporting prices will emerge. To this end we recall that the domain of the *adjoint* of a differential operator depends crucially on the choice of the domain for the differential operator [9, p. 1223]. With this in mind we are led to consider the subspace of W defined by

$$W_0=\{z\in W \mid \lambda z=0\}=\ker\lambda,$$

namely, the *kernel* of λ. Noting that W_0 is dense in Z, we form the adjoint L_0^* of the restriction of L to W_0, $L_0z=Lz$, $\forall z\in W_0$. By definition

$$\langle p, L_0z\rangle=\langle L_0^*p, z\rangle, \qquad \forall z\in W_0,$$

where L_0^* is well defined since W_0 is dense in Z [9, p. 1188]. It is easy to check that

$$L_0^*p=(-\dot{p},0), \qquad L_0^*\in\mathscr{L}(W_{\nu-1}^{\infty;n}, \mathscr{L}_{\nu-1}^{\infty;n+m}).$$

3.2 Definition. Let $L\in\mathscr{L}(W,V)$, $\lambda\in\mathscr{L}(W,Q)$, and let W be dense in Z. We say that the spaces (Z,W,V,Q) and the linear operators (L,λ) satisfy the *adjoint relation* $(\mathscr{R})$ if there exists a unique linear operator $\pi\in\mathscr{L}(Y,Q^*)$ such that

$$\langle p, Lz\rangle+\langle -L_0^*p, z\rangle+\langle \pi p, \lambda z\rangle=0, \qquad \forall z\in W, \quad \forall p\in Y, \tag{$\mathscr{R}$}$$

where $L_0^*\in\mathscr{L}(Y,Z^*)$ is the adjoint of the restriction of L to the kernel of λ and $Y=\{p\in V^* \mid L_0^*p\in Z^*\}$.

3.3 *Remark.* Lemma 3.1 implies that the spaces (3.4) and the linear operators (3.5) satisfy the adjoint relation $(\mathscr{R})$ with

$$\pi p=p(0), \qquad \pi\in\mathscr{L}(Y,R^n), \qquad Y=W_{\nu-1}^{\infty;n}.$$

$(-L_0^*,\pi)$ is the pair of linear operators in terms of which the dual problem $(\mathscr{P}^*)$ will be formulated, just as (L,λ) is the pair of linear operators used to formulate problem $(\mathscr{P})$.

It is convenient to let $\langle\!\langle\ ,\ \rangle\!\rangle$ denote the natural scalar product on $(Z^* \times V^* \times Q^*) \times (Z \times V \times Q)$. If we define

$$\beta(z) = (z, Lz, \lambda z), \qquad \gamma(p) = (-L_0^* p, p, \pi p), \tag{3.6}$$

then the adjoint relation ($\mathscr{R}$) may be written as

$$\langle\!\langle \gamma(p), \beta(z) \rangle\!\rangle = 0, \qquad \forall z \in W, \quad \forall p \in Y. \tag{$\mathscr{R}$}$$

This is the basic scalar product between prices and quantities in the analysis that follows.

3.4 *Remark.* The adjoint relation ($\mathscr{R}$) can be established for more general linear operators and spaces. For example if λ is surjective and $\ker \lambda$ is dense in Z (the so-called *trace property*) and if the spaces (Z, W, V, Q) are appropriate Hilbert (Sobolev) spaces, then ($\mathscr{R}$) is satisfied [14, Theorem 2.1, p. 114; 2, Theorem 1, p. 484]. The adjoint relation ($\mathscr{R}$) forms the key to the abstract approach to partial differential equations developed by Lions and others [14, 15].

4. Existence of Supporting Price

In this section I assume that the spaces (Z, W, V, Q) and the linear operators (L, λ) satisfy the adjoint relation ($\mathscr{R}$). To establish the existence of a supporting price I shall use the theory of pricing based on the concept of *conjugate convex functions.* To this end I reduce the convex programming problem ($\mathscr{P}$) to an unconstrained maximum problem in terms of a single function F. Let the function

$$F(\cdot): Z \times V \times Q \to [-\infty, \infty)$$

be defined by

$$\begin{aligned} F(y) = F(y_1, y_2, y_3) &= U(y_1) + \Psi_{\mathscr{G}}(y_1, y_2) + \Psi_{\mathscr{A}}(y_1) + \Psi_{\mathscr{B}}(y_3), \\ &\qquad\qquad \text{if} \quad y_1 \in W \\ &= -\infty, \qquad \text{if} \quad y_1 \notin W, \end{aligned} \tag{4.1}$$

where $\mathscr{G} = \{(y_1, y_2) \mid y_2 \in G(y_1)\}$ denotes the graph of $G(\cdot)$, and $\Psi_{\mathscr{G}}(\cdot)$,

$\Psi_{\mathscr{A}}(\cdot)$, and $\Psi_{\mathscr{B}}(\cdot)$ are the indicator functions of $\mathscr{G}$, $\mathscr{A}$, and $\mathscr{B}$ respectively, the *indicator function* of a subset S of a space X being defined by

$$\begin{aligned} \Psi_S(x) &= 0, && \text{if} \quad x \in S \\ &= -\infty, && \text{if} \quad x \notin S. \end{aligned}$$

If we use the notation (3.6), then (4.1) implies that $F(\beta(z))$ is defined on Z and $\bar{z}$ is an optimal program if and only if

$$F(\beta(\bar{z})) = \omega = \sup_{z \in Z} F(\beta(z)) = \sup_{z \in W} F(\beta(z)). \qquad (\mathscr{P})$$

The existence of a supporting price will be obtained by introducing a dual function defined as follows.

4.1 Definition. The function $F^*(\cdot)\colon Z^* \times V^* \times Q^* \to [-\infty, \infty)$ defined by

$$F^*(q) = \inf_{y \in Z \times V \times Q} [\langle\langle -q, y \rangle\rangle - F(y)]$$

will be called the *concave dual* of $F(\cdot)$.

4.2 Definition. If $\bar{y} \in \operatorname{dom} F$, then the subset

$$\partial F(\bar{y}) = \{\bar{q} \in Z^* \times V^* \times Q^* \mid F(y) - F(\bar{y}) \leqq \langle\langle \bar{q}, y - \bar{y} \rangle\rangle, \forall y \in Z \times V \times Q\}$$

is called the *superdifferential* of F at $\bar{y}$. F is said to be *superdifferentiable* at $\bar{y}$ if $\partial F(\bar{y}) \neq \varnothing$.

4.3 *Remark.* It follows from Definitions 4.1 and 4.2 that

$$-\bar{q} \in \partial F(\bar{y}) \qquad \textit{if and only if} \quad \bar{y} \textit{ maximises } F(y) + \langle\langle \bar{q}, y \rangle\rangle;$$

furthermore,

$$-\bar{q} \in \partial F(\bar{y}) \qquad \textit{if and only if} \quad F^*(\bar{q}) + F(\bar{y}) = \langle\langle -\bar{q}, \bar{y} \rangle\rangle. \qquad (4.2)$$

Since $F^*(\cdot)$ is defined on $Z^* \times V^* \times Q^*$, the function $F^*(\gamma(p))$ is defined on $Y = \{p \in V^* \mid L_0 p^* \in Z^*\}$. Consider the problem of finding $\bar{p} \in Y$ such that

$$F^*(\gamma(\bar{p})) = \omega^* = \sup_{p \in Y} F^*(\gamma(p)). \qquad (\mathscr{P}^*)$$

By the definition of $F^*(\cdot)$,

$$F^*(\gamma(p)) + F(\beta(z)) \leqq \langle\langle -\gamma(p), \beta(z) \rangle\rangle, \qquad \forall z \in W, \quad \forall p \in Y. \qquad (\mathscr{I})$$

($\mathscr{N}$), ($\mathscr{R}$), ($\mathscr{P}$), and ($\mathscr{P}^*$) imply

$$\omega^* + \omega \leqq 0.$$

4.4 Definition. $\bar{p} \in Y$ satisfying (i) $F^*(\gamma(\bar{p})) = \omega^*$ and (ii) $\omega^* + \omega = 0$ will be called a *supporting price.*

If we define $F^*(\gamma(p)) = -\infty$ for $p \in V^* \backslash Y$, then ($\mathscr{P}^*$) becomes

$$\omega^* = \sup_{p \in V^*} F^*(\gamma(p)) = \sup_{p \in Y} F^*(\gamma(p)). \qquad (\mathscr{P}^*)$$

4.5 *Remark.* It follows from (4.2) that *$\bar{z}$ is an optimal program and $\bar{p}$ is a supporting price if and only if*

$$-\gamma(\bar{p}) \in \partial F(\beta(\bar{z})). \qquad (\mathscr{E})$$

I shall now imbed problem ($\mathscr{P}$) into a family of *perturbed problems* along the lines of Rockafellar's approach to duality theory [10, Chapter III]. The key step is the introduction of the following function.

4.6 Definition. The function $\alpha(\cdot)$: $V \times Q \to [-\infty, \infty]$ defined by

$$\alpha(\xi) = \sup_{z \in Z} F(z, \Lambda z + \xi),$$

where $\Lambda z = (Lz, \lambda z)$ is called the *marginal function.*

The principal result of Rockafellar's abstract approach to the problem of pricing is the following equivalence criterion. The proof given by Ekeland and Temam [10, Proposition 2.2, p. 51] is readily adapted to the present context. A related result was obtained by Gale [11, Theorem 2, p. 23].

4.7 Theorem. *Let the spaces (Z, W, V, Q) and the linear operators (L, λ) satisfy the adjoint relation ($\mathscr{R}$). If $F(\cdot)$: $Z \times V \times Q \to [-\infty, \infty)$ is a concave, upper semicontinuous function, then there exists a supporting price if and only if the marginal function is superdifferentiable at $\xi = 0$.*

I shall derive a sufficient condition for the superdifferentiability of the marginal function by forming a perturbation of the feasible set $\mathscr{F}$ in (2.2).

4.8 Definition. For $\xi \in V \times Q$ let

$$\mathscr{F}_\xi = \{z \in \mathscr{A} \mid \xi \in K(z) - \Lambda z\}.$$

I shall say that the feasible set $\mathscr{F}$ is *nonempty under perturbations* if there exists a sphere $S_\eta(0)$ of radius $\eta > 0$ about the origin in $V \times Q$ such that

$$\mathscr{F}_\xi \neq \varnothing, \qquad \forall \xi \in S_\eta(0). \qquad (\mathscr{S})$$

The following condition for the lower semicontinuity of a correspondence is of basic importance. The result is due to Ursescu [26] and Robinson [21; 2, Theorem 1, p. 546].

4.9 THEOREM. *Let X and Y be Banach spaces, $R: X \to Y$ a correspondence whose graph is a closed-convex subset of $X \times Y$. If* $\operatorname{Int} \operatorname{dom} R^{-1} \neq \varnothing$, *then $R^{-1}(y)$ is lower semicontinuous for all $y \in$* $\operatorname{Int} \operatorname{dom} R^{-1}$.

This result leads directly to the following condition for the superdifferentiability of the marginal function.

4.10 THEOREM. *If $(U, \mathscr{F})$ are regular and $\mathscr{F}$ is nonempty under perturbations, then the marginal function is superdifferentiable at $\xi = 0$.*

Proof. Consider the correspondence $R(\cdot): W \to V \times Q$ defined by

$$\begin{aligned} R(z) &= K(z) - \Lambda z, && \text{if} \quad z \in \mathscr{A} \\ &= \varnothing, && \text{if} \quad z \notin \mathscr{A}. \end{aligned}$$

Clearly, $\mathscr{F}_\xi = R^{-1}(\xi)$. By Assumptions 2 and 3 the graph of R is a closed convex subset of $W \times V \times Q$. Since W, V, and Q are Banach spaces and since $(\mathscr{S})$ holds, by Theorem 4.9 $R^{-1}(\xi)$ is lower semicontinuous for all $\xi \in S_\eta(0)$. Since $\alpha(\xi) = \sup_{z \in R^{-1}(\xi)} U(z)$ and since by Assumption 1 $U(\cdot)$ is continuous, by the maximum theorem [4, Theorem 1, p. 115] $\alpha(\xi)$ is lower semicontinuous for all $\xi \in S_\eta(0)$. Since by Assumption 1 $U(\cdot)$ is concave and since R has a convex graph, $\alpha(\xi)$ is concave. It follows that $\alpha(\xi)$ is continuous for all $\xi \in S_\eta(0)$. Thus by a standard result on superdifferentiability [10, Proposition 5.2, p. 22] $\partial\alpha(0) \neq \varnothing$. ∎

Combining Theorems 4.7, 4.10, and Remark 4.5 yields the following.

4.11 THEOREM. *Let the spaces (Z, W, V, Q) and the linear operators (L, λ) satisfy the adjoint relation $(\mathscr{R})$. If $(U, \mathscr{F})$ are regular and $\mathscr{F}$ is nonempty under perturbations, then there exists a supporting price $\bar{p} \in Y$. If $\bar{z} \in W$ is an optimal program, then the pair $(\bar{z}, \bar{p})$ satisfies the global Euler–Lagrange inclusion*

$$-\gamma(\bar{p}) \in \partial F(\beta(\bar{z})). \qquad (\mathscr{E})$$

4.12 *Remark.* $(\mathscr{E})$ is the infinite dimensional version of the familiar Euler–Lagrange inclusion, which reduces to (E) in Section 5 when the spaces and linear operators are given by (3.4) and (3.5). $(\mathscr{E})$ can be expressed in a variety of equivalent ways. A form that is useful for the qualitative analysis of the equilibrium pair $(\bar{z}, \bar{p})$ is the Hamiltonian form. A variant of this form that reveals the role of the functions and sets that underlie ($\mathscr{P}$) is obtained as follows. We introduce the *return function* $\Phi: Z \times Y \to [-\infty, \infty)$ defined by

$$\Phi(z, p) = \sup_{v \in G(z)} \langle p, v \rangle. \tag{4.3}$$

Φ is concave in z and convex in p. Since the superdifferential of the indicator function $\Psi_S(x): X \to [-\infty, 0]$ of a convex subset S of a Banach space X is the *normal cone to S at x*,

$$\partial \Psi_S(x) = N_S(x) = \{q \in X^* \mid \langle q, \xi - x \rangle \geqq 0, \forall \xi \in S\}$$

and since $-(r, p) \in \partial \Psi_{\mathscr{G}}(z, v)$ if and only if $r \in -\partial_z \Phi(z, p)$, $v \in \underline{\partial}_p \Phi(z, p)$, where $\underline{\partial}_p \Phi(z, p)$ denotes the subdifferential of Φ with respect to p, applying the theorem for the superdifferential of a sum of functions [10, Proposition 5.6, p. 26] yields the following.

4.13 PROPOSITION. *If the pair $(\bar{z}, \bar{p}) \in W \times Y$ satisfies the global Euler–Lagrange inclusion ($\mathscr{E}$), then it satisfies the global Hamiltonian inclusions*

$$\begin{aligned} L_0^* \bar{p} &\in \partial_z \Phi(\bar{z}, \bar{p}) + \partial U(\bar{z}) + N_{\mathscr{A}}(\bar{z}) \\ Lz &\in \underline{\partial}_p \Phi(\bar{z}, \bar{p}) \\ -\pi \bar{p} &\in N_{\mathscr{B}}(\lambda \bar{z}). \end{aligned} \tag{$\mathscr{H}$}$$

5. PRICING PROGRAMS OVER AN INFINITE HORIZON

I shall now consider the abstract problem of resource allocation in the framework of continuous time over an infinite horizon. The technology and preferences will be characterised by *integral functionals* along the lines developed in [17, 18]. I shall show that the resulting problem can be posed as a special case of the convex programming problem studied in Sections 2 and 4, the basic Banach spaces and linear operators being those studied in Section 3.

I consider the economic activity of an economy over an infinite horizon. At each instant $t \in [0, \infty)$ a vector of *stocks* $x(t) = (x_1(t),..., x_n(t))$ of $n \geqq 1$ capital goods is used to produce a flow output $c(t) = (c_1(t),..., c_n(t))$ of *consumption* and $\dot{x}(t) = (\dot{x}_1(t),..., \dot{x}_n(t))$ of *investment*. It is convenient to let

$$z(t) = (x(t), c(t)), \qquad t \in [0, \infty).$$

I shall follow the ideas in [17] first constructing a feasible set of programs and then choosing a program space W that reflects the maximum growth capacity of these programs.

5.1 DEFINITION. A *technology correspondence* $\Gamma(t)$: $[0, \infty) \to R^{3n}$ is a set of triples (χ, ω, ζ) giving the consumption-investment flows (ω, ζ) producible with the stocks χ at time t. $\Gamma(\cdot)$ is said to be *regular* if it is a closed, convex valued measurable correspondence.

5.2 DEFINITION. For $x_0 \in R^n_+$ let $\mathscr{B} = \{\chi \in R^n_+ \mid \chi \leqq x_0\}$ denote the set of *feasible initial conditions.* Let $\mathscr{G}(x_0)$ denote the set of paths $(x, c, \dot{x}) \in \mathscr{M}^{3n}$, where x is absolutely continuous and

$$x(0) \in \mathscr{B}, \qquad (x(t), c(t), \dot{x}(t)) \in \Gamma(t) \quad \text{a.e.}$$

Let D be a closed convex subset of R^n_+ and let $A(t) = D \times R^n$ a.e., $\mathscr{A}' = \{z \in \mathscr{M}^{2n} \mid z(t) \in A(t) \text{ a.e.}\}$. The set of *technologically feasible paths* is given by

$$\mathscr{G}_{\mathscr{A}}(x_0) = \{(z, \dot{x}) \in \mathscr{G}(x_0) \mid z \in \mathscr{A}'\}.$$

5.3 DEFINITION. I shall say that Γ satisfies a *growth condition* if (i) $\mathscr{G}_{\mathscr{A}}(x_0) \neq \varnothing$ and (ii) there exists a growth function γ such that

$$\|z(t), \dot{x}(t)\| \leqq \gamma(t) \quad \text{a.e.} \qquad \forall (z, \dot{x}) \in \mathscr{G}_{\mathscr{A}}(x_0). \tag{5.1}$$

If Γ satisfies a growth condition with growth function γ and if ν is a density function relative to γ, then

$$\mathscr{G}_{\mathscr{A}}(x_0) \subset \mathscr{L}^{1,3n}_{\nu}.$$

If we introduce the *local indicator functions*

$$\psi_A: R^{2n} \times [0, \infty) \to [-\infty, 0], \qquad \psi_\Gamma: R^{3n} \times [0, \infty) \to [-\infty, 0]$$

defined by

$$\begin{aligned} \psi_A(\xi, t) &= 0, & &\text{if} \quad \xi \in A(t), & \psi_\Gamma(\eta, t) &= 0, & &\text{if} \quad \eta \in \Gamma(t) \\ &= -\infty, & &\text{if} \quad \xi \notin A(t), & &= -\infty, & &\text{if} \quad \eta \notin \Gamma(t) \end{aligned} \quad \text{a.e.}$$

and the *global indicator functions*

$$\Psi_{\mathscr{A}}: \mathscr{L}^{1,2n}_{\nu} \to [-\infty, 0], \qquad \Psi_{\mathscr{G}}: \mathscr{L}^{1,3n}_{\nu} \to [-\infty, 0]$$

defined by

$$\Psi_{\mathscr{A}}(z) = \int_0^\infty \psi_A(z(t), t)\, dt, \qquad \Psi_{\mathscr{G}}(v) = \int_0^\infty \psi_\Gamma(v(t), t)\, dt, \tag{5.2}$$

then

$$\mathscr{A} = \text{dom } \Psi_{\mathscr{A}}, \qquad \mathscr{G} = \text{dom } \Psi_{\mathscr{G}}.$$

Thus if we choose the *program space*

$$W = W_{v}^{1,n} \times \mathscr{L}_{v}^{1,n} \tag{5.3}$$

and let the spaces of investment and initial conditions be given by

$$V = \mathscr{L}_{v}^{1,n}, \qquad Q = R^{n}; \tag{5.4}$$

if the operator $\Lambda: W \to V \times Q$ is defined by

$$\Lambda z = (Lz, \lambda z) = (\dot{x}, x(0)), \qquad \forall z \in W \tag{5.5}$$

then the set of *technologically feasible programs* $\mathscr{F} \subset W$ is given by

$$\mathscr{F} = \{z \in \mathscr{A} \mid \lambda z \in \mathscr{B}, (z, Lz) \in \mathscr{G}\} = \{z \mid (z, \dot{x}) \in \mathscr{G}_{\mathscr{A}}(x_0)\}. \tag{5.6}$$

5.4 *Remark.* It is clear that $\mathscr{B}$ is a closed convex subset of R^n. To show that $\mathscr{A} \subset \mathscr{L}_{v}^{1,2n}$ and $\mathscr{G} \subset \mathscr{L}_{v}^{1,3n}$ are closed convex sets we note that if $A(\cdot)$ and $\Gamma(\cdot)$ are regular, then $\psi_A(\cdot, t)$ and $\psi_\Gamma(\cdot, t)$ are upper semicontinuous and concave on R^{2n} and R^{3n}, respectively. A standard argument based on Fatou's lemma shows that $\Psi_{\mathscr{A}}(\cdot)$ and $\Psi_{\mathscr{G}}(\cdot)$ are upper semicontinuous (in the norm topology) on $\mathscr{L}_{v}^{1,2n}$ and $\mathscr{L}_{v}^{1,3n}$, respectively, so that $\mathscr{A}$ and $\mathscr{G}$ are closed. The convexity of $\mathscr{A}$ and $\mathscr{G}$ follows from the concavity of $\Psi_{\mathscr{A}}(\cdot)$ and $\Psi_{\mathscr{G}}(\cdot)$. Thus Assumption 2 is satisfied. In view of the topology (3.1) on $W_{v}^{1,n}$ it is clear that (L, λ) satisfy Assumption 3.

I shall use the following two properties to show that the feasible set $\mathscr{F}$ is nonempty under perturbations.

5.5 Definition. The technology correspondence Γ is *separable* if there exists a correspondence $f(\cdot)$ satisfying $0 \in f(\chi, t)$, $\forall (\chi, t) \in D \times [0, \infty)$, and $\alpha \leqq 0$ such that

$$\Gamma(t) = \{(\chi, \omega, \zeta) \mid \omega + \zeta \in f(\chi, t), (\omega, \zeta) \geqq (0, \alpha\chi)\} \quad \text{a.e.} \tag{5.7}$$

5.6 Definition. If Γ is separable, a feasible program z is *normal* if $(c(t), \dot{x}(t)) > (0, \alpha\chi(t))$ a.e.

As in Section 2 I assume that choice among programs in W is determined by a preference ordering $\succsim$ which is representable by a *utility function* $U(\cdot)$. I now make the additional assumption that $U(\cdot)$ is an integral functional which depends only on consumption

$$U(c) = \int_0^\infty u(c(t), t)\, dt. \tag{5.8}$$

I shall invoke the following properties for the *instantaneous utility function u*.

5.7 DEFINITION. $u(\omega, t)$: $R^n \times [0, \infty) \to (-\infty, \infty)$ is said to be *regular* if $u(\cdot, t)$ is continuous and concave, dom $u(\cdot, t) = R^n$ and $u(\omega, \cdot) \in \mathscr{M}$, for all $t \in [0, \infty)$ and $\omega \in R^n$, respectively. u is said to be *increasing* if $u(\omega', t) > u(\omega, t)$, if $\omega' > \omega$ ($\omega_i' > \omega_i$, $i = 1,..., n$) for all $t \in [0, \infty)$.

5.8 DEFINITION. u satisfies a *growth condition* if

(i) there exists a density function μ such that

$$\|\partial_\omega u(\omega, t)\| \leqq \mu(t) \qquad \forall \omega \in R^n \quad \text{a.e.}, \tag{5.9}$$

(ii)
$$-\infty < \int_0^\infty u(0, t)\, dt < \infty. \tag{5.10}$$

5.9 *Remark.* In Definitions 5.7 and 5.8 I assume that $u(\cdot, t)$ is defined on all of R^n. The extension of $u(\cdot, t)$ off the non-negative orthant of R^n allows $U(\cdot)$ to be defined on all of W, a property that plays an important role in establishing the existence of prices. The essential economic restriction on u is (5.9) which is a uniform *bounded steepness* condition of the type considered by Gale [12, p. 7].

5.10 DEFINITION. I shall say that (u, Γ) satisfy *compatible growth conditions* if (i) u and Γ each satisfy growth conditions, (ii) there exists $\beta > 0$ such that $\beta\mu(t) \leqq \nu(t)$ a.e., where ν is a density function relative to γ.

5.11 *Remark.* Consider the correspondence g induced by Γ by the definition $\zeta \in g(\chi, \omega, t)$ if and only if $(\chi, \omega, \zeta) \in \Gamma(t)$ a.e. To express the local version of the Euler–Lagrange inclusion ($\mathscr{E}$) in Hamiltonian form I introduce the local version of function (4.3), the *return function* $\phi(\xi, \eta, t)$: $R^{3n} \times [0, \infty) \to [-\infty, \infty)$ defined by

$$\phi(\xi, \eta, t) = \sup_{\nu \in g(\xi, t)} \eta\nu. \tag{5.11}$$

5.12 THEOREM. *Let the technology and preferences be represented by the integral functionals* (5.2) *and* (5.8).

(i) *If* (u, Γ) *are regular and satisfy compatible growth conditions, then there exists an optimal program* $\bar{z} \in W_v^{1,n} \times \mathscr{L}_v^{1,n}$.

(ii) *If in addition* $x_0 \in \operatorname{Int} \operatorname{dom} D$, Γ *is separable, u is increasing, and* $\bar{z}$ *is normal, then there exists a supporting price* $\bar{p} \in W_{v^{-1}}^{\infty,n}$, $\bar{p}(t) \geqslant 0$ a.e.

(iii) *The pair* $(\bar{z}, \bar{p})$ *satisfies the local Hamiltonian inclusions*

$$\begin{gathered}(-\dot{\bar{p}}(t), 0) \in \partial_\xi \phi(\bar{z}(t), \bar{p}(t), t) + \partial_\omega u(\bar{c}(t), t) + N_D(\bar{x}(t)) \\ \dot{\bar{x}}(t) \in \underline{\partial}_\eta \phi(\bar{z}(t), \bar{p}(t), t) \quad \text{a.e.}\end{gathered} \tag{H}$$

and the transversality condition

$$\lim_{t \to \infty} \bar{p}(t)\, \bar{x}(t) = 0. \tag{T}$$

Proof. (i) Since u is regular, u is a normal concave integrand [23, Theorem 2E, p. 176]. Since u satisfies the growth conditions (5.9) and (5.10), and there exists $\lambda > 0$ such that $\lambda \mu(t) \leqq v(t)$ a.e., $U(\cdot)$ defined by (5.8) is continuous on $\mathscr{L}_v^{1,n}$ in the norm topology [23, Theorem 3L, p. 204]. Since the preferred sets $\{c \mid U(c) \geqq U(c')\}$ are convex, the weak and strong closures coincide [9, Theorem V.3.13, p. 422]. Thus the upper semicontinuity of $U(\cdot)$ in the norm topology on $\mathscr{L}_v^{1,n}$ implies the upper semicontinuity of $U(\cdot)$ in the weak $\sigma(\mathscr{L}_v^{1,n}, \mathscr{L}_v^{\infty,n})$ topology. Since Γ is regular and satisfies a growth condition, Proposition 7.5 in [17] implies that the set $\{(c, \dot{x}) \mid (x, c, \dot{x}) \in \mathscr{G}_{\mathscr{A}}(x_0)\}$ is weakly compact. Thus the projection $\mathscr{C} = \{c \mid (x, c, \dot{x}) \in \mathscr{G}_{\mathscr{A}}(x_0)\}$ is weakly compact and (i) follows.

(ii) Since Assumption 1, and by Remark 5.4, Assumptions 2 and 3 are satisfied, $(U, \mathscr{F})$ are regular. Since the spaces and linear operators (5.3)–(5.5) satisfy the adjoint relation ($\mathscr{R}$), to apply Theorem 4.11 it remains to show that $\mathscr{F}$ is nonempty under perturbations. We need to find $\eta > 0$ such that for all $\xi \in S_\eta(0) \subset \mathscr{L}_v^{1,n} \times R^n$ there exists $z \in \mathscr{A}$ satisfying $\Lambda z \in K(z) - \xi$. In view of the separability of Γ, if $\xi = (\zeta, \theta)$, this is equivalent to showing that the differential inclusion

$$\begin{gathered}\dot{x}(t) \in f(x(t), t) - c(t) - \zeta(t) \\ x(t) \in D \quad \text{a.e.}, \qquad x(0) \in \mathscr{B} - \theta\end{gathered} \tag{5.12}$$

has a solution $x \in W_v^{1,n}$ for all $(\zeta, \theta) \in S_\eta(0)$. Since $\bar{z}$ is normal, $\bar{z}$ is an optimal program for the technology set (5.7) if the lower bounds on consumption and investment are removed. For this modified technology set we may choose $c \in \mathscr{L}_v^{1,n}$ and in particular $c = -\zeta$. Since $x_0 \in \operatorname{Int} D$, there exists $\eta > 0$ such that $x_0 - \theta \in D$ for all $\theta \in R^n$ such that $\|\theta\| < \eta$. By the separability of Γ, $0 \in f(\chi, t)$ for all $(\chi, t) \in D \times [0, \infty)$. Thus $x(t) = x_0 - \theta$ a.e. is a solution of (5.12) for all $\xi \in S_\eta(0)$. Since $x \in W_v^{1,n}$, $\mathscr{F}$ is nonempty under perturbations. By Theorem 4.11 there exists a supporting price $\bar{p} \in W_{v^{-1}}^{\infty,n}$.

(iii) Let $H(\cdot): \mathscr{L}_v^{1,k} \to [-\infty, \infty)$ be a proper integral functional

$$H(y) = \int_0^\infty h(y(t), t)\, dt,$$

where $h(\cdot)$ is a normal concave integrand satisfying

$$h(\xi, t) \leqq \beta(t) + b\, \|\xi\|\, v(t), \qquad \forall \xi \in R^k \tag{5.13}$$

for some integrable function β and constant $b \geqq 0$. It follows from a measurable selection argument [10, Proposition 2.1, p. 271] that the concave dual of $H(\cdot)$ is

$$H^*(q) = \int_0^\infty h^*(q(t), t)\, dt,$$

where

$$h^*(\eta, t) = \inf_{\xi \in R^k} [-\eta\xi - h(\xi, t)] \quad \text{a.e.}$$

Thus $H^*(q) + H(y) + \langle q, y\rangle = 0$ if and only if

$$\int_0^\infty (h^*(q(t), t) + h(y(t), t) + q(t)\, y(t))\, dt = 0 \tag{5.14}$$

which is satisfied if and only if

$$h^*(q(t), t) + h(y(t), t) + q(t)\, y(t) = 0 \quad \text{a.e.}$$

since the integrand in (5.14) is nonpositive. Thus

$$-q \in \partial H(y) \qquad \text{if and only if} \quad -q(t) \in \partial h(y(t), t) \quad \text{a.e.} \tag{5.15}$$

Let $h = u + \psi_\Gamma + \psi_A$. Since u, ψ_Γ, and ψ_A are normal concave integrands, h is a normal concave integrand. Since u satisfies a growth condition and by virtue of the definition of an indicator function, h satisfies (5.13). By Theorem 4.11 the pair $(\bar{z}, \bar{p})$ satisfies the global Euler–Lagrange inclusion $(\mathscr{E})$ which reduces to

$$-(\dot{\bar{p}}, 0, \bar{p}) \in \partial H(\bar{z}, L\bar{z}), \qquad -\bar{p}(0) \in N_{\mathscr{B}}(\lambda\bar{z}). \tag{5.16}$$

(5.15) and (5.16) imply that the pair $(\bar{z}, \bar{p})$ satisfies the *local Euler–Lagrange inclusion*

$$-(\dot{\bar{p}}(t), 0, \bar{p}(t)) \in \partial h(\bar{z}(t), \dot{\bar{x}}(t), t) \quad \text{a.e.} \tag{E}$$

It follows from the separability of Γ and the definition of h that $\bar{p}(t) \in \partial u(\bar{c}(t), t)$ a.e. Since u is increasing $\bar{p}(t) \geqslant 0$ a.e. The argument used to show that $(\mathscr{E})$ implies $(\mathscr{H})$ shows that (E) implies (H), substituting ϕ in (5.11) for Φ in (4.3). Since $(\bar{x}, \bar{p}) \in W_v^{1,n} \times W_{v^{-1}}^{\infty,n}$, the transversality condition (T) follows at once from Lemma 3.1(i). ∎

5.13 *Remark.* Theorems 4.11 and 5.12 can also be used to analyse the problem of resource allocation when the economy consists of a finite number of producers with technology sets $(\mathscr{F}_1,..., \mathscr{F}_m)$ and a finite number of consumers with preference functions $(U_1,..., U_k)$.

Let $\Gamma_i(\cdot)$, $i = 1,..., m$, denote the technology correspondence of each firm, $D_i = R_+^n$, $i = 1,..., m$.

ASSUMPTION (a). *$\Gamma_i(\cdot)$ is regular and separable, $i = 1,..., m$, and $\Gamma(\cdot) = \sum_{i=1}^m \Gamma_i(\cdot)$ satisfies a growth condition, with growth function γ.*

Let $\hat{\mathscr{F}}_i$ denote the set of programs $(z_i, \dot{x}_i)$ such that $z_i \in W_v^{1,n} \times \mathscr{L}_v^{1,n}$, $(z_i(t), \dot{x}_i(t)) \in \Gamma_i(t)$, $x_i(t) \in R_+^n$ a.e., $x_i(0) = x_i^0$, $i = 1,..., m$. Then $\hat{\mathscr{F}} = \sum_{i=1}^m \hat{\mathscr{F}}_i$ and $\mathscr{F} = \{z \mid (z, Lz) \in \hat{\mathscr{F}}\}$.

I assume that the preference ordering of each consumer can be represented by an integral functional

$$U_j(c^j) = \int_0^\infty u_j(c^j(t), t)\, dt, \qquad j = 1,..., k.$$

5.14 DEFINITION. A utility function u is *strictly increasing* if $u(\omega', t) > u(\omega, t)$ whenever $\omega' \geqslant \omega$ $(\omega_i' \geqq \omega_i,\ i = 1,..., n,\ \omega_j' > \omega_j$ for some $j)$ for all $t \in [0, \infty)$. I say that U defined by (5.8) is (*strictly*) *increasing* if u is (strictly) increasing.

ASSUMPTION (b). *u_j is regular, strictly increasing, and satisfies a growth condition compatible with v, where v is a density function relative to γ, $j = 1,..., k$.*

Let $\Sigma^{k-1} = \{\alpha \in R_+^k \mid \sum_{j=1}^k \alpha_j = 1\}$ denote the $(k-1)$-dimensional simplex. Consider the family of *induced utility* (*convolution*) *functions* U: $\mathscr{L}_v^{1,n} \to (-\infty, \infty)$ defined by

$$U(c) = U(c; \alpha) = \sup_{c^j \in \mathscr{L}_v^{1,n}} \left[\sum_{j=1}^k \alpha_j U_j(c^j) \,\middle|\, \sum_{j=1}^k c^j = c \right], \qquad \alpha \in \Sigma^{k-1}. \quad (5.17)$$

For $c \in \mathscr{L}_v^{1,n}$, I let $c \geqslant 0$ denote $c(t) \geqslant 0$ a.e.

5.15 THEOREM. *Let Assumptions* (a) *and* (b) *be satisfied.*

(i) *For every $\alpha \in \Sigma^{k-1}$ there exists an optimal program $\bar{z} \in W_v^{1,n}$. If*

$\alpha > 0$, $\bar{z}$ induces a Pareto optimum $(\bar{z}_1,\ldots,\bar{z}_m,\bar{c}^1,\ldots,\bar{c}^k)$ where $\bar{z} = \sum_{i=1}^m \bar{z}_i$, $\bar{c} = \sum_{j=1}^k \bar{c}^j$.

(ii) *Let $x_0 = \sum_{i=1}^m x_i^0 > 0$ and let $(\bar{z}_1,\ldots,\bar{z}_m,\bar{c}^1,\ldots,\bar{c}^k)$ be a Pareto optimum. If $\bar{z}$ is normal, $\bar{x}(t) > 0$ a.e. and $\bar{c}^j \geqslant 0$, $j = 1,\ldots,k$, then there exists a price system $\bar{p} \in W_{\nu-1}^{\infty,n}$, $\bar{p}(t) > 0$ a.e. such that $[\bar{z}_1,\ldots,\bar{z}_m,\bar{c}^1,\ldots,\bar{c}^k,\bar{p}]$ is an equilibrium with transfer payments.*

Proof. (i) follows by a straightforward argument from Theorem 5.12(i). (ii) Let $\mathscr{C} = \{c \mid (x,c) \in \mathscr{F}\}$. Since $\bar{z}$ is normal we may assume $c \in \mathscr{C}$ implies $c \geqq c' \in \mathscr{C}$. Let $\mathscr{U} = \{(U_1,\ldots,U_k) \mid U_j = U_j(c^j), \sum_{j=1}^k c^j \in \mathscr{C}\}$. By the free disposal and convexity of $\mathscr{C}$ and since U_j are concave and increasing, $\mathscr{U}$ is a convex subset of R^k. By a standard separation theorem, since $(\bar{z}_1,\ldots,\bar{z}_m, \bar{c}^1,\ldots,\bar{c}^k)$ is a Pareto optimum there exists $\bar{\alpha} \in \Sigma^{k-1}$ such that

$$\sum_{j=1}^k \bar{\alpha}_j \bar{U}_j \geqq \sum_{j=1}^k \bar{\alpha}_j U_j, \qquad \forall (U_1,\ldots,U_k) \in \mathscr{U}, \tag{5.18}$$

where $\bar{U}_j = U_j(\bar{c}^j)$. Thus $\bar{z}$ is an optimal program for the utility function (5.17) with $\alpha = \bar{\alpha}$. Since by Assumption (b) U_j is concave and continuous on $\mathscr{L}_\nu^{1,n}$, $j = 1,\ldots,m$, it is readily shown that U is concave and continuous on $\mathscr{L}_\nu^{1,n}$. By the proof of Theorem 5.12, $\mathscr{F}$ is nonempty under perturbations. Thus by Theorem 4.11, $(\mathscr{E})$ is satisfied. Since $\bar{x}(t) > 0$ a.e., this implies

$$-(\dot{\bar{p}}, 0, \bar{p}) \in (0, \partial U(\bar{c}), 0) + \partial \Psi_{\mathscr{F}}(\bar{x}, \bar{c}, \dot{\bar{x}}). \tag{5.19}$$

If we let $\bar{q} = -\dot{\bar{p}}$, since $\hat{\mathscr{F}} \subset \mathscr{F}(x_0)$, (5.19) and the separability of Γ imply

$$\langle \bar{p}, \bar{c} + \dot{\bar{x}} \rangle - \langle \bar{q}, \bar{x} \rangle \geqq \langle \bar{p}, c + \dot{x} \rangle - \langle \bar{q}, x \rangle, \qquad \forall (x, c, \dot{x}) \in \hat{\mathscr{F}}$$

and hence, since $\hat{\mathscr{F}} = \sum_{i=1}^m \hat{\mathscr{F}}_i$,

$$\langle \bar{p}, \bar{c}_i + \dot{\bar{x}}_i \rangle - \langle \bar{q}, \bar{x}_i \rangle \geqq \langle \bar{p}, c_i + \dot{x}_i \rangle - \langle \bar{q}, x_i \rangle, \qquad \forall (x_i, c_i, \dot{x}_i) \in \hat{\mathscr{F}}_i, \quad i = 1,\ldots,m \tag{5.20}$$

so that each firm maximises intertemporal profit, $\langle \bar{p}, c_i + \dot{x}_i \rangle$ being the revenue derived from flow output and $\langle \bar{q}, x_i \rangle$ the rental cost of capital.

(5.19) and the separability of Γ imply $\bar{p} \in \partial U(\bar{c})$. It is readily shown that this implies $\bar{p} \in \bar{\alpha}_j \partial U_j(\bar{c}^j)$, $j = 1,\ldots,k$. By the definition of a supergradient

$$\bar{\alpha}_j(U_j(c^j) - U_j(\bar{c}^j)) \leqq \langle \bar{p}, c^j - \bar{c}^j \rangle, \qquad \forall c^j \in \mathscr{L}_\nu^{1,n}, \quad j = 1,\ldots,k. \tag{5.21}$$

Since $\bar{c}^j \geqslant 0$ and U_j is strictly increasing, $j = 1,\ldots,k$, $\bar{\alpha}$ in (5.18) satisfies $\bar{\alpha}_j > 0$, $j = 1,\ldots,k$. It follows from (5.21) that each consumer maximises utility subject to the budget constraint

$$\langle \bar{p}, c^j - \bar{c}^j \rangle \leqq 0, \qquad j = 1,\ldots,k.$$

This property combined with (5.20) and $\sum_{i=1}^{m} \bar{c}_i = \bar{c} = \sum_{j=1}^{k} c^j$ is the *definition* of an equilibrium with transfer payments $[\bar{z}_1,..., \bar{z}_m, \bar{c}^1,..., \bar{c}^k, \bar{p}]$. ∎

5.16 *Remark.* Since the equilibrium price satisfies $\bar{p} \in W^{\infty;n}_{v^{-1}}$, by Lemma 3.1

$$\langle \bar{p}, \dot{x}_i \rangle + \langle \dot{\bar{p}}, x_i \rangle = -\bar{p}(0)\, x_i(0), \qquad \forall x_i \in W^{1,n}_v.$$

Thus if $\mathscr{C}_i = \{c_i \mid (z_i, \dot{x}_i) \in \hat{\mathscr{F}}_i\}$, then (5.20) implies

$$\langle \bar{p}, \bar{c}_i \rangle \geqq \langle \bar{p}, c_i \rangle, \qquad \forall c_i \in \mathscr{C}_i, \quad i = 1,..., m.$$

Thus firms may also be viewed as purchasing new capital equipment at each instant at the cost $\bar{p}(t)\, \dot{x}_i(t)$ and subsequently owning the capital. The intertemporal profit which is maximised then reduces to $\langle \bar{p}, c_i \rangle$. Thus the *stock-flow-rental* economy of Theorem 5.15 which is characterised by the activities $[x_i, c_i, \dot{x}_i, c^j, i = 1,..., m, j = 1,..., k]$ and the price system $(\bar{q}, \bar{p})$ may be reduced to a *flow-ownership* economy characterised by the activities $[c_i, c^j, i = 1,..., m, j = 1,..., k]$ and the price system $\bar{p}$. In the latter description of the economy capital has disappeared and only the flow activity of producers and consumers remains.

5.17 *Remark.* Theorem 5.15(ii) is the familiar theorem of welfare economics [8, Theorem 2 and Remark] giving conditions under which a system of prices can be adjoined to a Pareto optimum to generate an equilibrium with transfer payments. Theorem 4.11 serves in place of the separation theorem, which is not applicable here, to guarantee the existence of a system of prices. Clearly, in the reduced form flow-economy the existence of $\bar{p} \in \mathscr{L}^{\infty,n}_v$ may be obtained via a separation theorem, by exploiting the lower semicontinuity of U.

5.18 *Remark.* [16] shows in a related framework how results similar to (i) and (ii) in Theorem 5.15 may be supplemented by a fixed point argument to obtain the existence of a competitive equilibrium.

6. Applications

In this section I consider two simple applications of the results of the previous section. In the first case feasible programs are *bounded*, in the second case feasible programs are *unbounded* and capable of growth at a positive exponential rate. In both cases I assume that the utility function U satisfies the following condition:

ASSUMPTION A. *The utility function u in* (5.8) *is given by*

$$u(\omega, t) = u(\omega)\, e^{-\delta t}, \qquad 0 < \delta < \infty,$$

where (i) $u(\omega)\colon R \to R$ *is regular,* (ii) *there exists* $b > 0$ *such that* $|\partial u(\omega)| \leqq b$ *for all* $\omega \in R$, (iii) u *is increasing.*

6.1 EXAMPLE. (Bounded case). Consider the following reformulation of the classical one commodity economy of Koopmans [13].

ASSUMPTION B. *The output correspondence f in* (5.7) *is given by*

$$f(\chi, t) = [0, h(\chi)] \ \text{ a.e.}, \qquad h(\chi) = \int_0^{\chi} (\phi(\xi) - \lambda)\, d\xi, \quad \chi \in R_+,$$

where (i) $\phi(\chi') \leqq \phi(\chi)$ *for all* $\chi' \geqq \chi \in R_+$, (ii) $0 < \lambda < \phi(0)$, (iii) *there exists* $\chi' \in (0, \infty)$ *such that* $\phi(\chi') < \lambda$.

$\phi(\chi)$ denotes the *marginal product* of capital and λ denotes its exponential *depreciation rate.* Assumption B implies that there exists $\hat{\chi} \in (0, \infty)$ such that $h(0) = h(\hat{\chi}) = 0$, $h(\chi) > 0$, $\chi \in (0, \hat{\chi})$. We let $D = [0, \hat{\chi}]$, $\alpha \in (-\infty, -\lambda]$. Let $m = |\alpha\hat{\chi}| + \max_{0 \leq \chi \leq \hat{\chi}} h(\chi)$. If $x_0 \in [0, \hat{\chi}]$, then every feasible path satisfies $x(t) \in [0, \hat{\chi}]$, $c(t) \in [0, m]$, $\dot{x}(t) \in [-m, m]$ a.e. Thus Γ satisfies a growth condition with

$$\gamma(t) = ae^{\tilde{\varepsilon} t} \ \text{ a.e.}, \qquad a = \max\{\hat{\chi}, m\}$$

for any $\tilde{\varepsilon} > 0$, so that

$$v(t) = ae^{-\varepsilon t} \ \text{ a.e.}, \qquad \varepsilon > \tilde{\varepsilon} \tag{6.1}$$

is a density function relative to γ. Since $\delta > 0$ by Assumption A, there exists $0 < \varepsilon < \delta$, and since by A(ii)

$$|\partial u(\omega) \cdot e^{-\delta t}| \leqq be^{-\delta t} = \mu(t) \ \text{ a.e.}, \qquad \forall \omega \in R \tag{6.2}$$

there exists $\beta > 0$ such that

$$\beta\mu(t) \leqq v(t) \ \text{ a.e.}$$

so that (u, Γ) satisfy compatible growth conditions. If we note that ϕ in (5.11) reduces to $\phi(\chi, \omega, \eta, t) = \eta(h(\chi) - \omega)$ a.e. with $\xi = (\chi, \omega)$ and that $N_D(\chi) = \{0\}$ if $\chi \in \operatorname{Int} D$, then Theorem 5.12 yields the following.

6.2 PROPOSITION. (i) *If Assumptions* A *and* B *are satisfied and* v *is given by* (6.1), *then there exists an optimal program* $\bar{z} \in W_v^{1,1} \times \mathscr{L}_v^{1,1}$.

(ii) *If in addition* $x_0 \in (0, \hat{\chi})$ *and* $\bar{z}$ *is normal, then there exists a supporting price* $\bar{p} \in W^{\infty, 1}_{\nu^{-1}}$, $\bar{p}(t) > 0$ a.e. *such that* (H, T) *are satisfied.*

(iii) *If* (u, h) *are differentiable and* $\bar{x}(t) \in (0, \hat{\chi})$ a.e., *then* (H, T) *reduce to*

$$\begin{aligned} -\dot{\bar{p}}(t) &= \bar{p}(t)\, h'(\bar{x}(t)), & 0 &= \bar{p}(t) - u'(\bar{c}(t))\, e^{-\delta t} \\ \dot{\bar{x}}(t) &= h(\bar{x}(t)) - \bar{c}(t) \quad \text{a.e.}, & \lim_{t \to \infty} &\bar{p}(t)\, \bar{x}(t) = 0. \end{aligned} \tag{6.3}$$

6.3 *Remark.* Equations (6.3) form the basis for Koopmans' analysis. Assumptions A and B, $x_0 \in (0, \hat{\chi})$ are those of Koopmans [13, pp. 261, 275] with three exceptions. First, I make no differentiability assumption on (u, h). (H) are thus *differential inclusions* as opposed to the *differential equations* in (6.3). Second, Koopmans assumes $\delta < g(0) - \lambda$. This ensures $\bar{x}(t) > 0$ a.e. and hence the validity of (6.3). Third, I am obliged to bound the marginal utility of consumption from above so that utility functions considered by Koopmans for which $u'(\omega) \to \infty$ as $\omega \to 0$ are excluded from my analysis.

6.4 Example. (Unbounded case). Consider a simple extension of Koopmans' economy to allow for two effects in production. First, the *absence of depreciation* of capital goods ($\lambda = 0$), $f(\chi, t) \to [0, \infty)$ as $\chi \to \infty$ for fixed t, and second, the effect of *technical change*, $f(\chi, t) \to [0, \infty)$ as $t \to \infty$ for fixed χ.

6.5 Definition. A function $h(\cdot): R_+ \to R_+$ is said to have an *asymptotic exponent* σ if for every $\alpha > 0$ there exists χ' such that

$$\chi^{\underline{\sigma}} < h(\chi) < \chi^{\bar{\sigma}}, \qquad \forall \chi > \chi',$$

where $\underline{\sigma} = \sigma - \alpha$, $\quad \bar{\sigma} = \sigma + \alpha$.

Assumption C. *The output correspondence* f *in* (5.7) *is given by*

$$f(\chi, t) = [0, e^{\theta t} h(\chi)] \quad \text{a.e.}, \qquad h(\chi) = \int_0^{\chi} \phi(\xi)\, d\xi, \quad \chi \in R_+,$$

where (i) $\phi(\chi') \leqq \phi(\chi)$ *for all* $\chi' \geqq \chi \in R_+$, (ii) $\phi(\chi) > 0$ *for all* $\chi \in R_+$, (iii) h *has an asymptotic exponent* $\sigma \in (0, 1)$, (iv) $\theta \in (0, \infty)$.

Let $D = [0, \infty)$, $\alpha = 0$. In the proof of Theorem 10.1 in [17], I showed that if we let $\rho = \theta/(1 - \sigma)$, then for every $\tilde{\varepsilon} > 0$ there exists $a_{\tilde{\varepsilon}} > 0$ such that every feasible path satisfies

$$\|z(t), \dot{x}(t)\| \leqq \gamma(t) = a_{\tilde{\varepsilon}} e^{(\rho + \tilde{\varepsilon})t} \quad \text{a.e.}$$

Thus

$$v(t) = a_{\tilde{\varepsilon}} e^{-(\rho+\varepsilon)t} \text{ a.e.,} \qquad \varepsilon > \tilde{\varepsilon} \tag{6.4}$$

is a density function relative to γ. If $\delta > \rho$, then there exists $\varepsilon > 0$ such that $\delta > \rho + \varepsilon$. Thus if μ is given by (6.2), there exists $\beta > 0$ such that $\beta\mu(t) \leqq v(t)$ a.e., so that (u, Γ) satisfy compatible growth conditions. Theorem 5.12 yields the following.

6.6 PROPOSITION. (i) *Let* $\rho = \theta/(1-\sigma)$. *If Assumptions* A *and* C *are satisfied,* $\delta > \rho$ *and* v *is given by* (6.4), *then there exists an optimal program* $\bar{z} \in W_v^{1,1} \times \mathscr{L}_v^{1,1}$.

(ii) *If in addition* $x_0 > 0$ *and* $\bar{z}$ *is normal, then there exists a supporting price* $\bar{p} \in W_{v^{-1}}^{\infty,1}$, $\bar{p}(t) > 0$ a.e. *such that* (H, T) *are satisfied.*

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